Complex Analysis: Resit Exam

Aletta Jacobshal 01, Friday 13 April 2018, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of each answer sheet and on the envelope.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope).
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

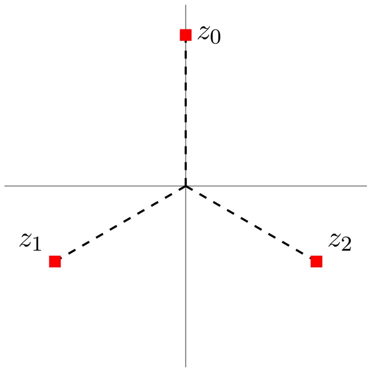
Question 1 (18 points)

(a) (3 points) Solve $z^3 + i = 0$ (you may express the solutions in Cartesian, trigonometric, or exponential form). Draw the solutions on the complex plane.

Solution

We have the equation $z^3 = -i = e^{3\pi i/2}$. Therefore, the solutions are

$$\begin{aligned} z_0 &= e^{\pi i/2} = i, \\ z_1 &= e^{\pi i/2}e^{2\pi i/3} = e^{7\pi i/6} = -e^{\pi i/6} = -\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}, \\ z_2 &= e^{7\pi i/6}e^{2\pi i/3} = e^{11\pi i/6} = e^{-\pi i/6} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}. \end{aligned}$$



(b) (15 points) Evaluate

$$\text{pv}\int_{-\infty}^{\infty} \frac{x}{x^3 + i}\,\mathrm{d}x$$

using the calculus of residues. *NB: This subquestion uses the result from subquestion (a) and will be graded **only if** subquestion (a) has been correctly answered and the result from subquestion (a) is properly used.*

Solution

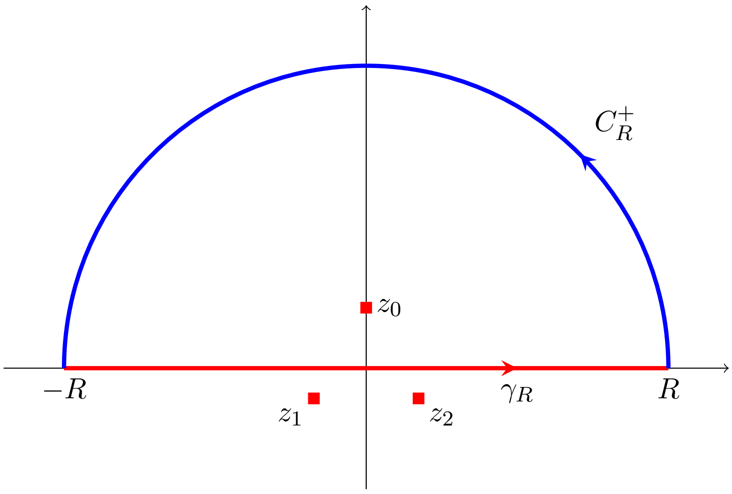
The integrand does not have any singularities along the real axis so, by definition,

$$I = \text{pv} \int_{-\infty}^{\infty} \frac{x}{x^3 + i} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x}{x^3 + i} dx = \lim_{R\to\infty} I_R.$$

To compute this integral we consider the closed contour

$$\Gamma_R = \gamma_R + C_R^+,$$

shown below.



We have

$$I_R = \int_{-R}^{R} \frac{x}{x^3 + i} dx = \int_{\gamma_R} f(z) dz,$$

where

$$f(z) = \frac{z}{z^3 + i}.$$

Therefore,

$$\int_{\Gamma_R} f(z) dz = I_R + \int_{C_R^+} f(z) dz.$$

For $R > 1$ we have

$$\int_{C_R} f(z) dz = 2\pi i \,\text{Res}(z_0) = \frac{2\pi}{3},$$

where we used that

$$\begin{aligned}
\text{Res}(z_0) &= \lim_{z\to i} (z - i) \frac{z}{(z - i)(z - z_1)(z - z_2)} = \frac{i}{(i - z_1)(i - z_2)} \\
&= \frac{i}{i^2 - i(z_1 + z_2) + z_1 z_2} = \frac{i}{(-1) - i(-2i \sin(\pi/6)) + (-1)} \\
&= \frac{i}{-2 - 2\sin(\pi/6)} = -\frac{i}{3}.
\end{aligned}$$

Moreover, since the degree of the numerator is 1 and the degree of the denominator is $3 \geq 1 + 2$ we have that

$$\lim_{R \to \infty} \int_{C_R^+} f(z)dz = 0.$$

Then taking the limit $R \to \infty$ we get

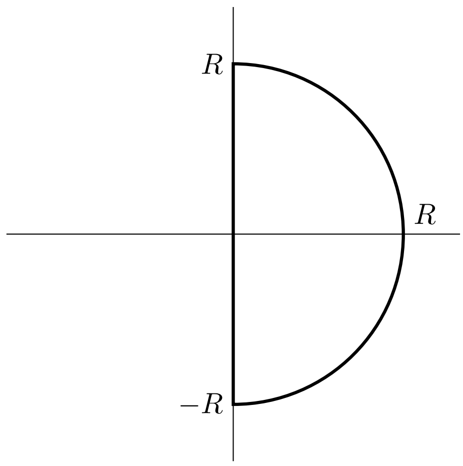
$$\frac{2\pi}{3} = I + 0,$$

giving

$$I = \frac{2\pi}{3}.$$

Question 2 (15 points)

Apply Rouché's theorem with the type of contour C_R shown below to prove that the equation $z = 2 - e^{-z}$ has exactly one solution in the right half plane (that is, for $\text{Re}(z) \geq 0$). Why must this solution be real?



Solution

Let $f(z) = z - 2$, $h(z) = e^{-z}$, and $g(z) = f(z) + h(z)$. Then the given equation becomes $g(z) = 0$.

To apply Rouché's theorem we first note that the functions $f(z)$ and $g(z)$ are analytic on and inside the given contour C_R.

The function $f(z)$ has a simple zero at $z = 2$. Thus for $R > 2$ the number of zeros of $f(z)$ inside C_R, counting multiplicity, is $N_0(f) = 1$.

Write $z = x + iy$. Then

$$|e^{-z}| = |e^{-x}e^{-iy}| = e^{-x}.$$

Along C_R we have $x \geq 0$ and thus

$$|h(z)| = |e^{-z}| \leq 1, \quad \text{for } z \in C_R.$$

Moreover, on C_R we have for $R > 4$ that

$$|f(z)| = |z - 2| \geq 2.$$

This implies that for any $R > 4$ we have that on C_R the inequality

$$|h(z)| \leq 1 < 2 \leq |f(z)|$$

holds and therefore, from Rouché's theorem, $N_0(g) = N_0(f) = 1$.

That is, the equation $g(z) = 0$ has exactly one solution inside C_R when $R > 4$. Therefore, no matter how large we make R we will always find just one solution inside C_R and we conclude that there is only a single solution of the equation in the right half plane.

The unique solution z_0 must be real. Given that

$$g(z_0) = z_0 - 2 + e^{-z_0} = 0,$$

we compute that

$$g(\overline{z_0}) = \overline{z_0} - 2 + e^{-\overline{z_0}} = \overline{z_0 - 2 + e^{-z_0}} = \overline{g(z_0)} = 0.$$

Therefore, $\overline{z_0}$ is also a solution. Since z_0 is in the right half plane, $\overline{z_0}$ is also in the right half plane. However, there is only one solution in the right half plane so we must have $z_0 = \overline{z_0}$ implying that z_0 is real.

Question 3 (15 points)

Represent the function

$$f(z) = \frac{z^2}{z-2},$$

(a) (8 points) as a Taylor series around 0 and give its radius of convergence;

Solution

We have

$$\begin{aligned}
\frac{z^2}{z-2} &= -\frac{1}{2} z^2 \frac{1}{1-(z/2)} \\
&= -\frac{1}{2} z^2 \Big(1 + \Big(\frac{z}{2}\Big) + \Big(\frac{z}{2}\Big)^2 + \Big(\frac{z}{2}\Big)^3 + \Big(\frac{z}{2}\Big)^4 + \cdots\Big) \\
&= -\frac{z^2}{2} - \frac{z^3}{2^2} - \frac{z^4}{2^3} - \frac{z^5}{2^4} - \frac{z^6}{2^5} + \cdots \\
&= \sum_{k=2}^{\infty} \Big(-\frac{1}{2^{k-1}}\Big) z^k,
\end{aligned}$$

where we used the geometric series which converges for $|z/2| < 1$, i.e., $|z| < 2$. The only singularity of $(z - i)/(z - 2)$ is at $z = 2$ which is at a distance $|z| = 2$ from 0. Therefore, the radius of convergence is 2.

(b) (7 points) as a Laurent series in the domain $|z| > 2$.

Solution
Since $|z| > 2$, that is $|2/z| < 1$, we have

$$\frac{z^2}{z-2} = \frac{z}{1-\frac{2}{z}} = z\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right)$$
$$= z + 2 + \frac{2^2}{z} + \frac{2^3}{z^2} + \cdots$$
$$= z + 2 + \sum_{k=1}^{\infty} \frac{2^{k+1}}{z^k}.$$

Question 4 (15 points)

Consider the function

$$f(z) = x + ax^2 - y^2 + iy + ibxy,$$

where $a, b \in \mathbb{R}$.

(a) (9 points) Find the values of a and b for which $f(z)$ is entire.

Solution
We check the Cauchy-Riemann equations for $u = x + ax^2 - y^2$ and $v = y + bxy$. We have

$$\frac{\partial u}{\partial x} = 1 + 2ax, \qquad \frac{\partial u}{\partial y} = -2y,$$
$$\frac{\partial v}{\partial x} = by, \qquad \frac{\partial v}{\partial y} = 1 + bx.$$

Then the Cauchy-Riemann equations imply

$$(2a - b)x = 0, \quad (b - 2)y = 0.$$

For f to be entire the two equations must be satisfied for all x, y implying $2a = b$ and $b = 2$. We conclude that

$$a = 1, \quad b = 2,$$

are the required values.

(b) (6 points) Let's call a_0 and b_0 respectively the (correct) values of a and b from subquestion (a). If $b = b_0$ but $a \neq a_0$ determine the subset of $\mathbb{C}$ where $f(z)$ is differentiable and the subset where it is analytic.

Solution
Since $b = 2$ the second Cauchy-Riemann equation is satisfied. However, for $b = 2$ the first equation becomes $2(a - 1)x = 0$ and since $a \neq 1$ it is satisfied only for $x = 0$.
This means that f is differentiable on the set $\{x + iy : x = 0, \, y \in \mathbb{R}\}$ which is the imaginary axis.
A line does not contain any open (in $\mathbb{C}$) subsets and therefore the function is nowhere analytic (the subset where f is analytic is the empty set).

Question 5 (12 points)

Consider the function

$$f(z) = \frac{\cos(z) + 1}{(z - \pi)^2}.$$

(a) (9 points) Compute the Laurent series of $f(z)$ for $|z - \pi| > 0$.

Solution

The Taylor series for $\cos(z)$ at $z = \pi$ is

$$\begin{aligned}
\cos(z) &= \cos(\pi + (z - \pi)) \\
&= \cos(\pi)\cos(z - \pi) - \sin(\pi)\sin(z - \pi) \\
&= -\cos(z - \pi) \\
&= -\Big(1 - \frac{1}{2!}(z - \pi)^2 + \frac{1}{4!}(z - \pi)^4 + \cdots\Big) \\
&= -1 + \frac{1}{2!}(z - \pi)^2 - \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k)!}(z - \pi)^{2k}.
\end{aligned}$$

Therefore, the Laurent series of the given function is

$$\frac{\cos(z) + 1}{(z - \pi)^2} = \frac{1}{2!} - \frac{1}{4!}(z - \pi)^2 + \cdots = \frac{1}{2!} - \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k)!}(z - \pi)^{2k-2}.$$

(b) (3 points) Determine the singularities of $f(z)$ and their type (removable, pole, essential; if pole, specify the order).

Solution

There is only a singularity at π. Since the Laurent series centered at π has only non-negative powers of $(z - \pi)$ we conclude that π is a removable singularity.

Question 6 (15 points)

(a) (6 points) Find the most general harmonic polynomial $ax^2 + 2bxy + cy^2$, that is, determine the relations between the (real) constants a, b, c, so that the given polynomial is a harmonic function.

Solution

We compute the Laplacian of u:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2a + 2c.$$

For a harmonic function the Laplacian should be 0, so we conclude that $c = -a$.
Therefore, the most general harmonic polynomial of the given form is

$$a(x^2 - y^2) + 2bxy.$$

(b) (9 points) Determine the harmonic conjugate of the polynomial $ax^2 + 2bxy + cy^2$ for the a, b, c determined in subquestion (a). *NB: Finding a harmonic conjugate will only work if the correct relations between a, b, c have been determined. If finding the harmonic conjugate does not work you should check the correctness of your result in subquestion (a).*

Solution

Alternative 1. We can observe that

$$z^2 = (x^2 - y^2) + 2ixy.$$

This implies $\text{Re}(z^2) = x^2 - y^2$ and $\text{Re}(-iz^2) = 2xy$.
Therefore, we consider the analytic function

$$f(z) = az^2 + b(-iz^2) = (a - ib)z^2.$$

We compute

$$f(z) = (a - ib)((x^2 - y^2) + 2ixy) = \big[a(x^2 - y^2) + 2bxy\big] + i\big[-b(x^2 - y^2) + 2axy\big].$$

The real part of the analytic function gives the harmonic function $a(x^2 - y^2) + 2bxy$ and thus its imaginary part gives the requested harmonic conjugate which is

$$-b(x^2 - y^2) + 2axy.$$

Alternative 2. Let $u = a(x^2 - y^2) + 2bxy$ and let v be the requested harmonic conjugate. Then u and v must satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},$$

giving

$$\frac{\partial v}{\partial y} = 2ax + 2by, \quad \frac{\partial v}{\partial x} = 2ay - 2bx.$$

Integrating the first equation with respect to y we obtain

$$v(x, y) = 2axy + by^2 + g(x).$$

Then substitution into the second equation gives

$$2ay + g'(x) = 2ay - 2bx \Rightarrow g(x) = -bx^2 + C.$$

Therefore,

$$v(x, y) = 2axy + by^2 - bx^2 + C,$$

where C can be set to any value.